# Integrals Associated with the Potts Model

(*Mathematica* Noteboook Slides)

Victor Adamchik
Carnegie Mellon University, USA

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## Introduction

Percolation theory is concern with fluid flow in random media, for example, molecules penetrating a porus solid or wildfires consuming a forest

Broadbent and Hammersley (1957) wondered about the probable number of open channels in media for fluid passage.

Let $M = (m_{i\,j})$ be a random binary matrix - the value of $m_{i\,j}$ is 1 if the probability $p$ and it is 0 if the probability is $1 - p$.

We introduce *s-cluster* as an isolated group of adjacent (horizontally or vertically) ones. The following matrix (aqssociated with a *site* percolation model) has 1 one-cluster, 2 two-clusters and 1 four-clusters:

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

The total number of clusters is 4.

What is the number of cluster when we consider *n x n* matrix?

This is also called the mean cluster density.

Considering random matrixes and using Monte Carlo simulation, Ziff showed that the limit when $n \to \infty$ exists.

Consider another kind of matrix associated with a *bond* percolation model

$$M = \begin{pmatrix} & 1 & & 0 & & 0 & \\ 1 & & 0 & & 0 & & 0 \\ & 0 & & 1 & & 0 & \\ 0 & & 1 & & 1 & & 0 \\ & 0 & & 1 & & 0 & \\ 1 & & 0 & & 0 & & 0 \\ & 0 & & 0 & & 0 & \end{pmatrix}$$

It has 1 one-cluster,1 two-clusters and 1 four-clusters.

Again, when *n* goes to infinity, the limit does exist.

Moreover, the limit can be expressed in an analytic form ( $y = \frac{\pi}{3}$)

$$\frac{d}{d\,y}\,\frac{1}{y}\int_{-\infty}^{\infty}\operatorname{sech}\left(\frac{\pi\,x}{2\,y}\right)\log\left(\frac{\cosh(x)-\cos(2\,y)}{\cosh(x)-1}\right)d\,x$$

Temperley && Lieb, 1971

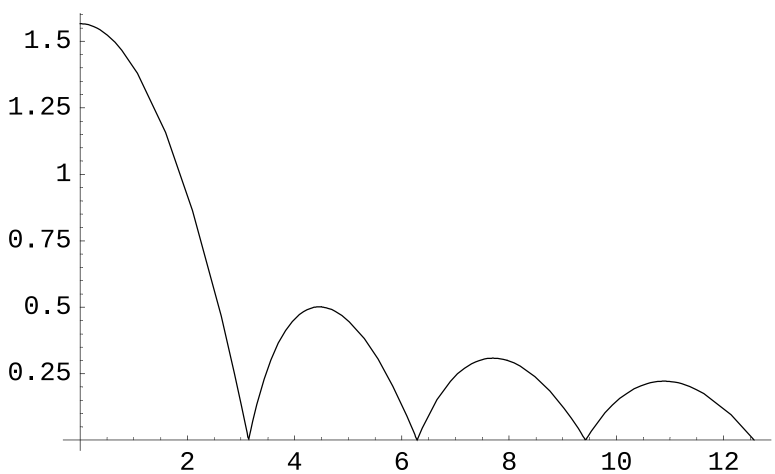
Define, for our purposes,

$$P_2(y) = \frac{1}{2\,y}\int_0^{\infty}\operatorname{sech}\left(\frac{\pi\,x}{2\,y}\right)\log\left(\frac{\cosh(x)-\cos(2\,y)}{\cosh(x)-1}\right)dx$$

Observe that the integral can be rewritten in the following alternative form (after performing integration by parts):

$$P_2(y) = \frac{4\sin^2(y)}{\pi}\int_0^{\infty}\frac{\tan^{-1}\left(\tanh\left(\frac{\pi\,x}{4\,y}\right)\right)\coth\left(\frac{x}{2}\right)}{\cosh(x)-\cos(2\,y)}\,dx$$

Here is a graph of $P_2(y)$ on the interval $y \in (0,\ 4\,\pi)$




Another way to investigate the structure of the percolation model is to consider the *mean cluster size*

For the first matrix, the mean is

$$\frac{1+2+2+4}{4}$$

For second

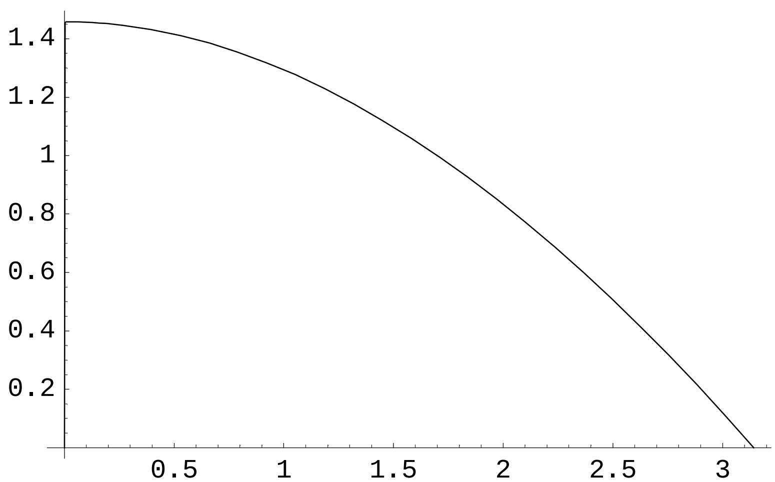
$$\frac{1+2+4}{3}$$

Again, the closed form cab be derived

Let us investigate what is known for triangular lattices.
Baxter, Temperley , Lieb (1978)

$$\frac{d}{d\,y}\int_0^{\infty}\frac{\sinh((\pi-y)\,x)\,\sinh\left(\frac{2\,y\,x}{3}\right)}{x\,\sinh(\pi\,x)\,\cosh(y\,x)}\,dx$$

The integral converges for $y \in [0,\ 3\,\pi)$. Here is its pix on an interval $y \in (0,\ \pi)$:




Just as the first derivative has an interpretation as a mean value, the second derivative can be regarded as a measure of "fluctuations" in the number of clusters corresponding to the bond percolation model.

The Potts model encompasses a number of problems in statistical physics and lattice theory. It generalizes the Ising model so that each spin can have more than two values. It includes the ice-vertex and bond percolation models as special cases. It also is related to graph-coloring problems: if we color each vertex in a lattice with one of q colors, in how many ways do we have exactly n pairs of adjacent vertices colored alike?

see
F. Y. Wu, *The Potts model*, Reviews of Modern Physics, 54 (1982) 235-268.
for a survey
.

## Square lattice

the limiting mean cluster density at the critical probability $\frac{1}{2}$:

$$K_b\left(\frac{1}{2}\right) = -\frac{1}{16} - \frac{\csc(y)}{4} \left.\frac{\partial W(y)}{\partial y}\right|_{y=\frac{\pi}{3}}$$

$$\lim_{n\to\infty} \frac{\log W(z)}{n} = \frac{1}{4y}\int_{-\infty}^{\infty} \operatorname{sech}\left(\frac{\pi x}{2y}\right) \log\left(\frac{\cosh(x)-\cos(2y)}{\cosh(x)-1}\right) dx$$

I will show that

$$K_b\left(\frac{1}{2}\right) = \frac{3\sqrt{3}}{2} - \frac{41}{16}$$

***Proof.***
By differentiation we obtain

$$\frac{8\sqrt{3}}{\pi}\int_0^{\infty} \frac{\tan^{-1}(\tanh(\frac{3x}{4}))\sinh(x)}{(2\cosh(x)+1)^2}\,dx - \frac{27}{\pi^2}\int_0^{\infty} x\cosh\left(\frac{x}{2}\right)\operatorname{csch}(3x)\,dx$$

The second integral is almost trivial

$$\frac{27}{\pi^2}\int_0^{\infty} x\,\mathrm{Cosh}\left[\frac{x}{2}\right]\mathrm{Csch}[3\,x]\,dx$$

$$\frac{27}{18+9\sqrt{3}}$$

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FullSimplify[%]
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$6-3\sqrt{3}$

In the first integral, we change the variable of integration $x \to -2\log(z)$.

$$\int_0^\infty \frac{\tan^{-1}(\tanh(\frac{3x}{4}))\sinh(x)}{(2\cosh(x)+1)^2}\,dx = \int_0^1 \frac{z(z^4-1)\tan^{-1}\left(\frac{z^3-1}{z^3+1}\right)}{(z^4+z^2+1)^2}\,dz$$

Remarkably this integral can be computed by using the Risch algorithm

$$\frac{8\sqrt{3}}{\pi}\int_0^1 \frac{z(z^4-1)\tan^{-1}\left(\frac{z^3-1}{z^3+1}\right)}{(z^4+z^2+1)^2}\,dz = -\sqrt{3}\left(-2+\sqrt{3}\right)$$

which completes the proof.

The second derivative can be regarded as a measure of "fluctuations" in the number of clusters corresponding to the bond percolation model.

We will prove here that

$$\left(\cot(y)\frac{d}{\text{dy}}\right)^2 P_2(y)\Big|_{y=\frac{\pi}{3}} = -\frac{1}{2}\left(25-8\sqrt{3}\right)+\frac{18}{\pi}$$

$$P_2(y) = \frac{1}{2y}\int_0^\infty \text{sech}\left(\frac{\pi x}{2y}\right)\log\left(\frac{\cosh(x)-\cos(2y)}{\cosh(x)-1}\right)dx$$

which, again, is a new result.

***Proof***. First of all, we observe that

$$\left(\cot(y)\frac{d}{\text{dy}}\right)^2 P_2(y)\Big|_{y=\frac{\pi}{3}} = -\frac{20}{3}+4\sqrt{3}+\frac{1}{3}\frac{\partial^2 P_2(y)}{\partial y^2}\Big|_{y=\frac{\pi}{3}} \qquad (1)$$

Differentiating the integral and changing the variable of integration $x \to -2\log(z)$, we arrive at

$$\left.\frac{\partial^2 P_2(y)}{\partial y^2}\right|_{y=\frac{\pi}{3}} = -\frac{16}{\pi}\int_0^1 \frac{z\,(z^8+7\,z^6-7\,z^2-1)\tan^{-1}\left(\frac{z^3-1}{z^3+1}\right)}{(z^4+z^2+1)^3}\,dz +$$

$$\frac{144\sqrt{3}}{\pi^2}\int_0^1 \frac{z^4\,(1-z^2)\log(z)}{(z^4-z^2+1)\,(z^4+z^2+1)^2}\,dz +$$

$$\frac{648}{\pi^3}\int_0^1 \frac{z^4\log(z)}{(z^2-1)\,(z^8+z^4+1)}\,dz - \frac{972}{\pi^3}\int_0^1 \frac{z^4\log^2(z)}{(z^2+1)\,(z^4-z^2+1)^2}\,dz$$

These integrals are NOT elementary, but could be evaluted in terms of di-logarithms which are usually defined as

$$\sum_{k=1}^{\infty}\frac{z^k}{k^2}$$

As we see derivatives of the above integral do not represent a problem.
But about the integral itself??

$$P_2(y) = \frac{1}{2\,y}\int_0^{\infty}\operatorname{sech}\left(\frac{\pi\,x}{2\,y}\right)\log\left(\frac{\cosh(x)-\cos(2\,y)}{\cosh(x)-1}\right)dx$$

- ***y* = 0**

Changing the variable of integration $x \to \frac{2\,y\,z}{\pi}$ and expanding the integrand at y = 0, we get:

$$P_2(0) = \frac{2}{\pi}\int_0^1 \frac{\log(\log^2(x)+\pi^2)}{x^2+1}\,dx -$$

$$\frac{4}{\pi}\int_0^1 \frac{\log(\log(\frac{1}{x}))}{x^2+1}\,dx$$

Both integrals can be found in Gradshteyn and Ryzhik, *Table of Integrals, Series and Products*:

$$\int_0^1 \frac{\log(\log(\frac{1}{x}))}{x^2+1}\,dx = \frac{\pi}{2}\log\left(\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\sqrt{2\,\pi}\right)$$

$$\int_0^1 \frac{\log(\log^2(x)+\pi^2)}{x^2+1}\,dx = \pi\log\left(\frac{\sqrt{\frac{\pi}{2}}\;\Gamma(\frac{1}{4})}{2\,\Gamma(\frac{3}{4})}\right)$$

Therefore,

$$P_2(0) = 4\log\left(\frac{\Gamma(\frac{1}{4})}{2\,\Gamma(\frac{3}{4})}\right)$$

## Triangular lattice

The mean cluster size corresponds to

$$P_3(y) = 3\int_0^\infty \frac{\sinh((\pi - y)\,x)\,\sinh\left(\frac{2\,y\,x}{3}\right)}{x\,\sinh(\pi\,x)\,\cosh(y\,x)}\,dx$$

Let $y = \frac{\pi\,p}{q}$ , where $p$ and $q$ are positive integers.

Then the intedgral can be rewritten as

$$P_3\,(y) = 3\int_0^1 \frac{z^{-p-1}\,(1 - z^{2\,p})\,(z^{3\,q} - z^{3\,p})}{(1 + z^{3\,p})\,(1 - z^{3\,q})}\,\frac{dz}{log\,z}$$

This integral belongs to the more common class of integrals

$$\int_0^1 \log\left(\log\left(\frac{1}{z}\right)\right)\frac{z^\alpha}{Q(z)^\beta}\,dz$$

where $\alpha$ and $\beta$ are positive integers and $Q(z)$ is a cyclotomic polynomial.

This class of integrals is the main topic!!

## The dimer model

a perfect matching of a graph is a set of edges that covers very vertex once

this is somewhat generalization to domino tilings of a chessboard

the dimer model is a set of such matching for infinite graphs

Kenyon, Baxter showed that a partition function associated with some kind of planar graph is

$$\int_{-\infty}^\infty \frac{\sin((\pi - r)\,x)\,\sinh(\frac{4\,r}{\pi}\,\theta\,x)}{x\,\sinh(\pi\,x)\,\cosh(r\,x)}\,dx$$

In his 2003 paper, Kenyon wrote "any information about this integral would be greately appreciated."

# Evaluation

as we saw, the above integrals can be converted to the form

$$\int_0^1 \frac{P(x)}{Q(x)} \log\log\left(\frac{1}{x}\right) dx$$

that is not known if these integrals can be evaluated in finite terms

There a few such integrals in Gradzhteyn & Ryzhyk and in Vardi's paper

I.Vardi, Integrals, an Introduction to Analytic Number Theory, *Amer. Math. Monthly*, 95 (1988).

however, if $Q(x)$ is a cyclotomic polynomial !!!

we can reduce the problem of integration to the following two classes of integrals

$$\int_0^1 \frac{x^{p-1}}{(1+x^n)^q} \log\log\left(\frac{1}{x}\right) dx$$

$$\int_0^1 x^{p-1} \left(\frac{1-x}{1-x^n}\right)^q \log\log\left(\frac{1}{x}\right) dx$$

**Theorem 1** Re($p$) > 0 and Re($n$) > 0

$$\int_0^1 \frac{x^{p-1}}{1+x^n} \log\log\left(\frac{1}{x}\right) dx =$$

$$\frac{\gamma + \log(2\pi)}{2n} \left(\psi\left(\frac{p}{2n}\right) - \psi\left(\frac{n+p}{2n}\right)\right) +$$

$$\frac{1}{2n} \left(\zeta'\left(1, \frac{p}{2n}\right) - \zeta'\left(1, \frac{n+p}{2n}\right)\right)$$

Here $\zeta$ is the Hurwitz zeta function

$$\zeta(z, a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^z}$$

*Proof*

We start with

$$\int_0^1 \frac{x^{p-1}}{x^n + \lambda}\, dx = \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k}{\lambda^k\,(k\,n + p)}$$

differentiating wrt to $p$

$$\int_0^1 \frac{x^{p-1}}{x^n + \lambda} \log^{q-1}\left(\frac{1}{x}\right) dx = \frac{\Gamma(q)}{\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k}{\lambda^k\,(k\,n + p)^q}$$

differentiating wrt to $q$

$$\int_0^1 \frac{x^{p-1}}{x^n + \lambda} \log\log\left(\frac{1}{x}\right) dx = -\gamma \int_0^1 \frac{x^{p-1}}{x^n + \lambda}\, dx - \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k \log(k\,n + p)}{\lambda^k\,(k\,n + p)}$$

compute a limit when $\lambda \to 1$. QED

When the above integral can be done in elementary functions?

**Corrolary 1**

$$\int_0^1 \frac{x^{n-1}}{1 + x^n} \log\log\left(\frac{1}{x}\right) dx = -\frac{\log(2)\log(2\,n^2)}{2\,n}$$

**Particular Cases:**

$$\int_0^1 \frac{1}{1 + x^2} \log\log\left(\frac{1}{x}\right) dx = \frac{\pi}{2} \log\left(\frac{\sqrt{2\,\pi}\;\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right)$$

$$\int_0^1 \frac{1}{1 - x + x^2} \log\log\left(\frac{1}{x}\right) dx = \frac{\pi\left(5\log(2\,\pi) - 6\log\Gamma(\frac{1}{6})\right)}{3\sqrt{3}}$$

What if we change the denominator a bit??

$$\int_0^1 \frac{1}{1+x+x^2} \log\log\left(\frac{1}{x}\right) dx$$

$$\int_0^1 \frac{1-x}{1-x^3} \log\log\left(\frac{1}{x}\right) dx$$

**Theorem 2.**

$$\int_0^1 x^{p-1} \frac{1-x}{1-x^n} \log\log\left(\frac{1}{x}\right) dx =$$

$$\frac{\gamma + \log n}{n} \left(\psi\left(\frac{p}{n}\right) - \psi\left(\frac{1+p}{n}\right)\right) +$$

$$\frac{1}{n} \left(\zeta'\left(1, \frac{p}{n}\right) - \zeta'\left(1, \frac{1+p}{n}\right)\right)$$

**Particular Cases:**

$$\int_0^1 \frac{x}{1+x+x^2+x^3+x^4+x^5} \log\log\left(\frac{1}{x}\right) dx$$

back to theorem 1.

it is possible to generalize it for higher orders...

**Theorem 3** Re($p$) > 0 and Re($n$) > 0

$$\int_0^1 \frac{x^{p-1}}{(1+x^n)^3} \log\log\left(\frac{1}{x}\right) dx$$

*involves the following trancendentals*

$$\zeta'\left(1, \frac{p}{2n}\right) - \zeta'\left(1, \frac{n+p}{2n}\right)$$

$$\zeta'\left(0, \frac{p}{2n}\right) - \zeta'\left(0, \frac{n+p}{2n}\right)$$

$$\zeta'\left(-1, \frac{p}{2n}\right) - \zeta'\left(-1, \frac{n+p}{2n}\right)$$

Generally speaking

**Theorem 4** Re($p$) > 0 and Re($n$) > 0

$$\int_0^1 \frac{x^{p-1}}{\left(1+x^n\right)^q} \log\log\left(\frac{1}{x}\right) dx$$

*is a linear combinations of*

$$\zeta'\left(1, \frac{p}{2n}\right) - \zeta'\left(1, \frac{n+p}{2n}\right)$$

$$\zeta'\left(0, \frac{p}{2n}\right) - \zeta'\left(0, \frac{n+p}{2n}\right)$$

$$\zeta'\left(-k, \frac{p}{2n}\right) - \zeta'\left(-k, \frac{n+p}{2n}\right)$$

where $k = 1, 2, \ldots, q-2$

but what is $\zeta'(-k, \frac{p}{2n})$ ???

The paper

B.Gosper, $\int_{\frac{n}{4}}^{\frac{m}{6}} \log\Gamma(z)\, dz$, *Amer. Math. Soc*, **14**(1997).

in which he showed that $\zeta'(-1, z)$ is related to the double Gamma function

$$\log G(x) = (x-1)\log\Gamma(x) + \zeta'(-1) - \zeta'(-1, x),$$

$$\mathrm{Re}(x) > 0$$

where (Barnes, 1900)

$$G(z+1) = \Gamma(z)\, G(z)$$

$$G(1)=1$$

a generalization of the functional equation for the Euler Gamma function

$$\Gamma(z+1) = z\,\Gamma(z)$$

$$\Gamma(1)=1$$

The Euler gamma is usually defined by the integral

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx$$

$$G(n+1) = \prod_{k=1}^{n} \Gamma(k) = \prod_{k=1}^{n-1} k!$$

but what we know about $\zeta'(-2, z)$ or $\zeta'(-3, z)$ and so on ???

they related to the multiple Gamma function

## Multiple gamma and the Hurwitz functions

The multiple zeta function is defined by (Barnes, 1900)

$$\zeta_n(s, z) = \sum_{k_1=0}^{\infty} \ldots \sum_{k_n=0}^{\infty} \frac{1}{(k_1 + k_2 + \ldots + k_n + z)^s}$$

which can be rewritten as a single sum (Vardi)

I. Vardi, Determinants of Laplacians and multipe gamma functions, *SIAM J. Math. Anal.*, **19**(1988), 493-507.

$$\zeta_n(s, z) = \sum_{k=0}^{\infty} \frac{1}{(k+z)^s} \binom{k+n-1}{n-1}$$

Vardi expressed the $\Gamma_n$ function in terms of $\zeta_n(s, z)$

$$(-1)^n \log \Gamma_n(z) = -\lim_{s \to 0} \left( \frac{\partial \zeta_n(s, z)}{\partial s} \right) - \sum_{k=1}^{n} (-1)^k \binom{z}{k-1} R_{n+1-k} \qquad (2)$$

where

$$R_n = \sum_{k=1}^{n} \lim_{s \to 0} \left( \frac{\partial \zeta_k(s, 1)}{\partial s} \right) \qquad (3)$$

**Proposition.** *The multiple gamma function $\Gamma_n(z)$ may be expressed by means of the derivatives of the Hurwitz zeta function:*

$$(-1)^n \log \Gamma_n(z) = \frac{1}{(n-1)!} \sum_{k=0}^{n-1} P_{k,n}(z) \left( \zeta'(-k) - \zeta'(-k, z) \right) \tag{4}$$

*where the polynomials* $P_{k,n}(z)$ *are defined by*

$$P_{k,n}(z) = \sum_{j=k+1}^{n} (-z)^{j-k-1} \binom{j-1}{k} \left[ {n \atop j} \right] \tag{5}$$

*where* $\zeta'(t, z) = \frac{d}{dt} \zeta(t, z)$ *and* $\left[ {n \atop i} \right]$ *are the Stirling cycle numbers.*

The polynomials $P_{k,n}(z)$ can be viewed as the generalized Stirling polynomials of the first kind, generated by

$$\prod_{k=1}^{n-1} (x + k - z) = \sum_{k=0}^{n-1} P_{k,n}(z)\, x^k$$

For $z = 1$ the polynomials $P_{k,n}(z)$ are simplified to the Stirling numbers:

$$P_{k,n}(1) = \left[ {n-1 \atop k} \right]$$

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Table[P[z, k, 5], {k, 0, 4}] // TableForm
```

$24 - 50\,z + 35\,z^2 - 10\,z^3 + z^4$
$50 - 70\,z + 30\,z^2 - 4\,z^3$
$35 - 30\,z + 6\,z^2$
$10 - 4\,z$
$1$

The functional equation

$$P_{k,n+1}(z) - n\,P_{k,n}(z) - P_{k,n+1}(z+1) = 0$$

$$P_{k,n}(z) = 0, \;\; k \geq n$$

Alternative forms

$$P_{k,n}(z) = \frac{(-1)^k}{k!} \frac{\partial^{n-1}}{\partial y^{n-1}} \frac{\log^k(1-y)}{(1-y)^{1-z}} \Bigg|_{y \to 0} \tag{6}$$

and

$$P_{k,n}(z) = \sum_{i=k+1}^{n} \binom{z}{n-i} \frac{(n-1)!}{(i-1)!} \left[ {i \atop k+1} \right] \tag{7}$$

Particular cases

$$P_{0,n}\left(\frac{1}{2}\right) = \frac{(2n-3)!!}{2^{n-1}}$$

$$P_{1,n}\left(\frac{1}{2}\right) = \frac{(2n-3)!!}{2^{n-2}} \sum_{k=0}^{n-2} \frac{1}{2k+1}$$

D. S. Mitrinovic, Sur une classe de nombres reliés aux nombres de Stirling, *C. R. Acad. Sci. Paris*, **252**(1961), 2354-2356.

Mitrinovic, D. S.; Mitrinovic, R. S., Tableaux d'une classe de nombres relies aux nombres de Stirling. *Univ. Beograd. Publ. Elektrotehn., Fak. Ser. Mat. Fiz.*, No. 77, 1962, 77 pp.

$$\prod_{k=0}^{n-1} (x-k-z) = \sum_{k=0}^{n} R_n^k(z)\, x^k$$

$$\prod_{k=1}^{n-1} (x+k-z) = \sum_{k=0}^{n-1} P_{k,n}(z)\, x^k$$

$$R_n^k(1) = \left[ {n \atop k} \right]$$

$$R_n^k(1-z) - R_n^k(-z) = n\,(-1)^{n-k}\, P_{k,n}(z)$$

# Inverse

we are mostly interested in the inverse, namely derivatives of the Hurwitz zeta as a function of the $\Gamma_k$:

$$\zeta'(-n) - \zeta'(-n, z) = F\left(\log \Gamma_1(z), \log \Gamma_2(z), \ldots, \log \Gamma_{n+1}(z)\right)$$

For example, if $n = 2$ we have

$$\zeta'(-2, z) - \zeta'(-2) = 2 \log \Gamma_3(z) + (3 - 2z) \log \Gamma_2(z) + (1 - z)^2 \log \Gamma(z)$$

**Proposition.** *The derivatives of the Hurwitz zeta function may be expressed by means of the multiple gamma function* $\Gamma_n(z)$

$$\zeta'(-n) - \zeta'(-n, z) = (-1)^n \sum_{k=0}^{n} k!\, Q_{k,n}(z) \log \Gamma_{k+1}(z) \tag{8}$$

*where the polynomials* $Q_{k,n}(z)$ *are defined by*

$$Q_{k,n}(z) = \sum_{j=k}^{n} (1-z)^{n-j} \binom{n}{j} \left\{ {j \atop k} \right\} \tag{9}$$

*and* $\left\{ {n \atop k} \right\}$ *are he Stirling subset numbers, defined by*

$$\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\},$$

$$\left\{ {n \atop 0} \right\} = \delta_{0,n}$$

For $z = 0$

$$Q_{k,n}(0) = \sum_{j=k}^{n} \binom{n}{j} \left\{ {j \atop k} \right\} = \left\{ {n+1 \atop k+1} \right\}$$

For $z = 1$

$$Q_{k,n}(1) = \left\{ {n \atop k} \right\}$$

```
Collect[Table[Q[k, 5, z], {k, 0, 5}], z] // TableForm
```

$$
\begin{array}{l}
1-5z+10z^2-10z^3+5z^4-z^5 \\
31-75z+70z^2-30z^3+5z^4 \\
90-125z+60z^2-10z^3 \\
65-50z+10z^2 \\
15-5z \\
1
\end{array}
$$

The functional equation

$$Q_{k,n+1}(z)-(k+1-z)\,Q_{k,n}(z)-Q_{k-1,n}(z) = 0$$

$$Q_{k,n}(z)=0,\ \ k \geq n$$

Shifted argument

$$Q_{k,n}(1-z)=\sum_{j=0}^{n}(z-1)^j\binom{n}{j}\left\{ {n-j+1 \atop k+1} \right\}$$

**Lemma.** *Polynomials* $P_{j,n}(z)$ *and* $Q_{k,j}(z)$ *satisfy the following discrete orthogonality relation:*

$$\sum_{j=k}^{n-1}(-1)^{j-k}\,Q_{k,j}(z)\ P_{j,n}(z) = \delta_{k,n-1} \tag{10}$$

*where* $\delta_{k,n-1}$ *is the Kronecker delta.*

The proof of Lemma is based on the discrete orthogonality relation for the Striling numbers:

$$\sum_{j=0}^{n}(-1)^{m+j}\left[ {j \atop m} \right]\left\{ {n \atop j} \right\} = \delta_{m,n}$$

$$2^{n-3}\,Q_{3,n}\left(\frac{1}{2}\right)=\frac{1}{48}\left(7^n-3\,5^n+3^{n+1}-1\right)$$

$$\left.\frac{\partial^2 Q_{2,n}(1-z)}{\partial z^2}\right|_{z\to 0}=2\binom{n}{n-2}\left\{ {n-2 \atop 2} \right\}$$